Finite Element Analysis of Slow Non–Newtonian Channel Flow

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A finite element method is applied to isothermal slow channel flow of power-law fluids. The fully developed flow is normal to the channel cross section. The method and results are compared with a finite difference method for rectangular channels and with exact solutions for the Newtonian case. An advantage of finite element methods is the flexibility of the mesh of elements approximating the continuum, chosen to suit the particular problem. Arbitrary boundary shapes can be handled as illustrated by a rectangular channel with rounded corners.

SCOPE

There are many industrial examples of slow non-Newtonian channel flows, such as in polymer processing which involves the flow of highly viscous melts in extruders, dies, and molds (for example, Bernhardt, 1959). In fact the work described here was prompted by the authors' interests in polymer processing (Fenner, 1970; Palit, 1969) although the methods and results are generally applicable. In slow flows (sometimes called creeping flows) fluid inertia forces are negligible compared with viscous and pressure forces.

Attention is confined to fully developed flows normal to the cross sections of locally uniform channels whose rigid boundaries may move in the direction of flow. Initially a rectangular channel with one moving boundary is treated, which is a simplification of the situation in a screw extruder, and for which analytical, finite difference and experimental results exist. The flow is assumed to be isothermal in the sense that temperature variations do not significantly affect the viscous properties of the fluid.

The object of the work described here is to apply numerical finite element (FE) methods to non-Newtonian flow problems traditionally solved by finite difference (FD) methods. The latter involve substituting FD approximations for the derivatives contained in the differential conservation equations. These approximations are in terms of the values of the variables at discrete points in the problem region. The points usually lie in rows parallel to the coordinate axes and are frequently, though not necessarily (Gosman et al., 1969) equispaced along these rows. Hence the method is readily applicable to the rectangular channel flow problem, as shown by Middleman (1965), Martin (1969), Dyer (1969), and Fenner (1970).

FE methods require the problem region to be divided into small subregions or finite elements. For example, a two-dimensional region may be divided into a mesh of triangles or quadrilaterals. The relevant variables are again required in terms of values at discrete nodal points in the mesh, the corners of the elements (and sometimes additional points on the element boundaries). An appropriate (polynomial) function of position is assumed for each variable within each element. A variational formulation (for example, Schechter, 1967) is normally used to solve for the nodal point variables by minimizing appropriate functionals, although in some cases, such as the present problem, an equivalent direct equilibrium formulation is also possible. FE methods are well established in the fields of structural and solid continuum mechanics (see Zienkiewicz, 1971), but have not been widely used in solving fluid mechanics and heat transfer problems: examples include Martin (1968), Oden and Somogyi (1969), Atkinson et al. (1969, 1970), and Tay and De Vahl Davis (1971). The main difference is that in a fluids application the elements represent spatial rather than material subregions of the continuum.

CONCLUSIONS AND SIGNIFICANCE

The applicability of a FE method to the analysis of slow non-Newtonian channel flows is demonstrated. The numerical results obtained for rectangular channels are successfully compared with both FD results and analytical solutions for Newtonian fluids. The real power of FE methods lies in their ability to handle nonuniform meshes, with elements concentrated in regions of large velocity gradients. A procedure is given for obtaining such meshes for rectangular channels which give very substantial improvements in numerical accuracy for a given number of nodal points. Thus fewer nodal points can be used with a corresponding reduction in computing times. The FE method can also be applied to flow in channels of arbitrary cross section with a minimum of additional labor. As an illustration results are obtained for a rectangular channel with rounded corners.

FD methods are mainly useful for analyzing flow in simple geometries which may be described by one of the major coordinate systems. FE methods, however, possess greater flexibility which allows them to be applied to much more complex situations with a minimum of adaptation for each new problem. Against this must be set, however, the greater initial computer programming effort. Nevertheless FE methods are potentially very powerful tools for the solution of engineering problems of fluid flow.

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FORMULATION OF THE PROBLEM

Figure 1 shows the rectangular channel geometry and coordinates, the third coordinate z being normal to this cross section. The top boundary moves with a downstream (z-direction) velocity V_z, while the other boundaries are fixed. For steady incompressible downstream flow the differential stress balance equation (derived from momentum conservation) is

$$P_z = \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial x} \tag{1}$$

where τ_{yz} and τ_{zx} are the viscous shear stress components and P_z is the locally constant downstream pressure gradient.

For a Stokesian fluid

$$\tau_{yz} = \mu \frac{\partial w}{\partial y}, \quad \tau_{zx} = \mu \frac{\partial w}{\partial x} \tag{2}$$

where $w(x, y)$ is the downstream velocity (the velocity components in the x and y directions are zero), and the viscosity is a function only of the second invariant of the rate of deformation tensor

$$\mu = \mu(I_2), \quad 4I_2 = \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \tag{3}$$

Although any functional form of $\mu(I_2)$ can in principle be handled, attention is confined to a power-law form which is applicable to many non-Newtonian fluids, particularly polymer melts (Fenner, 1970):

$$\mu = \mu_0 \left(\frac{\sqrt{4I_2}}{\gamma_0}\right)^{n-1} \tag{4}$$

where μ_0 is the effective viscosity at the reference shear rate γ_0, and n is the power-law index. Given the no slip boundary conditions

$$w(0, y) = w(B, y) = w(x, 0) = 0; \quad w(x, H) = V_z \tag{5}$$

the problem can be solved by FD methods (for example, Fenner, 1970; Gosman et al., 1969; and Middleman, 1965). Fenner (1970) took advantage of the symmetry of the flow and considered only half of the channel, and the FE formulation could do likewise.

The number of variable parameters can be reduced by dimensional analysis. The results take the form of flow curves relating P_z and the volumetric flowrate

$$Q = \int_0^B \int_0^H w \cdot dy \cdot dx \tag{6}$$

for various V_z, H, B and viscous properties. Dimensionless flowrate and pressure gradient can be defined as

$$\pi_Q = Q/BHV_z \tag{7}$$

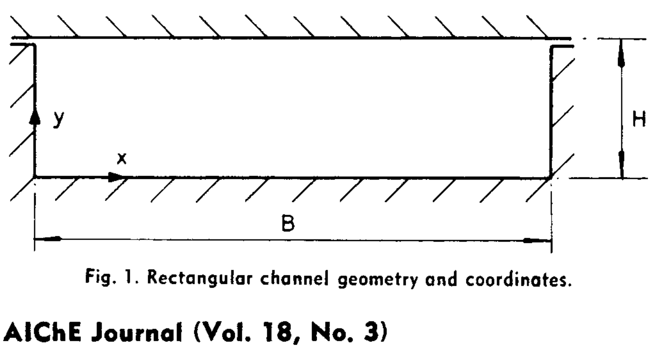

Fig. 1. Rectangular channel geometry and coordinates.

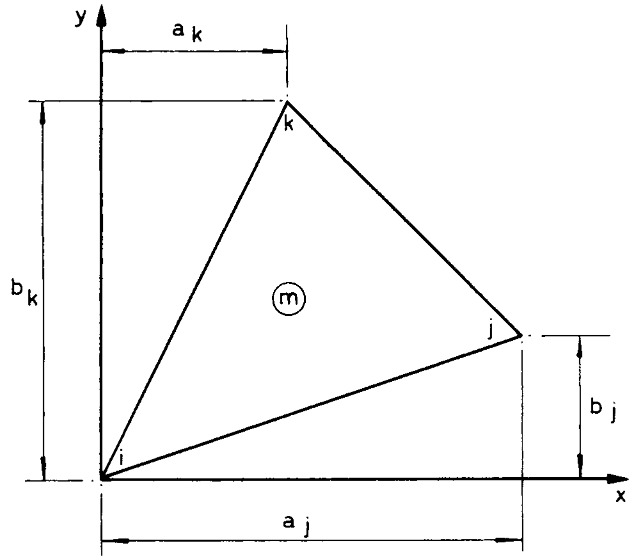

Fig. 2. Triangular element dimensions.

$$\pi_p = P_z H/\overline{\tau} \tag{8}$$

where $\overline{\tau}$ is the shear stress at the mean shear rate V_z/H

$$\overline{\tau} = \mu_0 \gamma_0 \left(\frac{V_z}{\gamma_0 H}\right)^n \tag{9}$$

The dimensionless flow curve takes the functional form

$$\pi_Q = \pi_Q (\pi_p, n, S) \tag{10}$$

where $S = H/B$ is the channel shape parameter. The coordinates and velocity may be nondimensionalized as

$$X = x/H, \quad Y = y/H, \quad W = w/V_z \tag{11}$$

hence

$$\pi_Q = S \int_0^{1/S} \int_0^1 W \cdot dY \cdot dX \tag{12}$$

Numerous FE methods are possible, involving elements of various shapes and complexities (see Zienkiewicz, 1971), but only one method is considered in detail here. A mesh of triangular elements with nodal points at the corners is selected as being simple to use. For non-Newtonian flow problems this has the advantage that the viscosity is constant over each element, greatly simplifying the analysis. For the present problem there are two alternative FE formulations, both yielding the same final equations.

Direct Equilibrium Formulation

The direct equilibrium FE formulation involves the construction of force balance equations for each nodal point, and is sometimes called the stiffness approach, which reflects its origin in structural analysis (see Palit, 1969; Wilson, 1963; and Zienkiewicz, 1971). Figure 2 shows a typical triangular element numbered m in the x-y plane (local coordinates parallel to the channel coordinates in Figure 1). The downstream velocity is assumed to vary linearly over the element

$$w = C_1 + C_2 x + C_3 y \tag{13}$$

where the three constants may be found in terms of the three nodal point velocities

$$C_1 = w_i, \quad [C_2\ C_3]^T = \frac{1}{2\Delta_m} [d]\ [w_i\ w_j\ w_k]^T \tag{14}$$

where

$$[d] = \begin{bmatrix} (b_j - b_k) & b_k & -b_j \\ (a_k - a_j) & -a_k & a_j \end{bmatrix}$$

and $\Delta_m = \frac{1}{2}(a_j b_k - a_k b_j)$ is the area of the element. Hence the shear rates are constant over the element

$$\frac{\partial w}{\partial x} = C_2, \quad \frac{\partial w}{\partial y} = C_3 \tag{15}$$

so that I_2, μ and consequently the shear stresses are also constant. Taking the element to be of unit downstream thickness the pressure and viscous stresses may be resolved into downstream forces acting at the nodal points. The pressure forces are

$$[s_{pi}^{(m)}\ s_{pj}^{(m)}\ s_{pk}^{(m)}]^T = -\frac{1}{3} P_z \Delta_m\ [1\ 1\ 1]^T \tag{16}$$

where, for example, $s_{pi}^{(m)}$ means the pressure force at point i due to element m, and the viscous forces are

$$[s_{vi}^{(m)}\ s_{vj}^{(m)}\ s_{vk}^{(m)}]^T = \frac{1}{2} [d]^T\ [\tau_{zx}\ \tau_{yz}]^T$$

$$= \frac{\mu_m}{4\Delta_m} [d]^T [d]\ [w_i\ w_j\ w_k]^T$$

$$= [k^{(m)}]\ [w_i\ w_j\ w_k]^T \tag{17}$$

where $[k^{(m)}]$ is the symmetric 3×3 viscous stiffness matrix for the element. When similar relations have been constructed for all the elements, the equilibrium condition for each nodal point can be specified by summing the component forces at that point and equating to zero [thus balancing pressure and viscous forces, analogous to Equation (1)]. For point i, say $(1 \leq i \leq N)$

$$\Sigma_m\ (s_{pi}^{(m)} + s_{vi}^{(m)}) = 0 \tag{18}$$

where the summation need only be performed for elements which involve the point i. The final result is a set of N equations which may be nondimensionalized as

$$[F] = [K]\ [W] \tag{19}$$

where $[F]$, $[K]$, $[W]$ are the dimensionless pressure force, stiffness and velocity matrices, with the elements of $[F]$ proportional to π_p.

The boundary conditions may be imposed, say for nodal point number q where the dimensionless velocity is specified as W_b by setting

$$F_q = W_b \cdot M, \quad K_{qq} = M \tag{20}$$

where M is a number very much larger than the nondiagonal elements of $[K]$.

The equations can only be solved with the aid of a digital computer. Although for Newtonian fluids solutions can be obtained by inverting the stiffness matrix, an iterative successive overrelaxation method (Varga, 1962) is preferable for large matrices. For non-Newtonian fluids an iterative method is essential, since after a prescribed number of iteration cycles for the velocities the viscosity of each element must be updated using the current shear rates to find I_2 and μ from Equations (3) and (4).

Variational Formulation

The variational FE formulation involves the minimization of the integral over the flow field of the appropriate functional, which here is the rate of dissipation of total potential energy (Schechter, 1967)

$$F = 2\mu I_2 - wP_z \tag{21}$$

The integral to be minimized with respect to the nodal point velocities is

$$\chi = \iint F \cdot d(\text{area}) \tag{22}$$

which is the sum of the integrals over the individual elements. The contribution of the typical element of Figure 2, with the linear velocity profile (13) is

$$\chi^{(m)} = \frac{1}{2}\mu_m \Delta_m\ (C_2^2 + C_3^2) - 1/3 P_z \Delta_m\ (w_i + w_j + w_k) \tag{23}$$

Hence, differentiating with respect to say w_i

$$\frac{\partial \chi^{(m)}}{\partial w_i} = \mu_m \Delta_m \left(C_2 \cdot \frac{\partial C_2}{\partial w_i} + C_3 \cdot \frac{\partial C_3}{\partial w_i} \right) - 1/3 P_z \Delta_m \tag{24}$$

The condition to be satisfied for every point i is

$$\frac{\partial \chi}{\partial w_i} = 0 = \Sigma_m \frac{\partial \chi^{(m)}}{\partial w_i} \tag{25}$$

where the summation need only be performed for elements which involve the point i.

The final set of N equations after nondimensionalization is identical to Equation (19). Such a result is to be expected because the variational principle expresses the equilibrium condition.

Choice of Mesh, Accuracy, and Convergence

Figure 3 shows part of the type of FE mesh used. Both nodal points and elements are numbered (in this case there are 81 points and 128 elements). In terms of computer programming it is convenient to start from a uniform mesh of this form, for which the nodal point numbers and coordinates are readily generated. Once generated the mesh can then be modified to suit the particular problem. Elements can be concentrated in regions of rapid variation of the velocity field, and the whole boundary shape can be changed as described below.

The distribution of points in the uniform mesh of Figure 3 is identical to that used in traditional FD methods. Under the right conditions the resulting equations are also identical. Consider, for example, mesh point 12. The five point FD equation for w_{12} involves four surrounding variables w_{11}, w_{21}, w_{13} and w_3 (Fenner, 1970), as does the present FE formulation (that is, the variables associated with nodes of triangles which include the point 12), and the equations are identical. On the other hand the FE equation for w_{11} involves eight surrounding variables, and the mesh is said to be irregular (Zienkiewicz, 1971). This

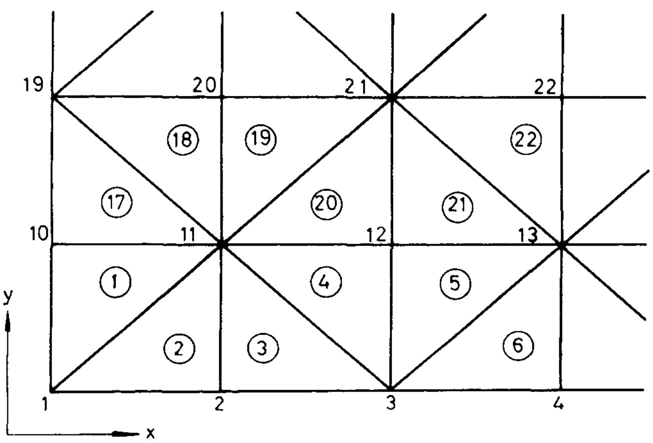

Fig. 3. Part of an 81 (9 × 9) point FE mesh.

mesh is chosen as being both a simple pattern and one which ensures that the corner points are associated with more than one element (for example, an element with points 1, 2, 10 would have a uniformly zero velocity profile). Indeed, the inclusion of the corner points is perhaps the major difference between the FE and FD methods for this uniform mesh. Because of the similarity of the equations it is to be expected that the present FE formulation which assumes linear variations over triangular elements is of the same order of accuracy as the five point FD method.

When solving equations by successive overrelaxation it is important to consider whether the process is convergent. That is, whether as the number of iterations is increased the variables tend to constant values. The only formal proofs of convergence currently available are for linear systems of equations, and require that $[K]$ should be diagonally dominant (Varga, 1962). This means that every term on the diagonal should be at least equal in magnitude to the other terms in the same row of the matrix. For the present problem (flow normal to the plane of the mesh) this condition holds for any mesh of acute or right angled triangles. While this is a sufficient condition for convergence it is not always necessary: the inclusion of some obtuse angled triangles may be permissible.

Thus convergence is assured for a Newtonian fluid with the present mesh, and experience suggests that it is not seriously endangered by the introduction of a shear dependent viscosity although the rate of convergence is reduced. It is also affected by the frequency with which the element viscosities are updated, although all the results presented here are for updating after every velocity iteration cycle.

Results have been obtained without convergence difficulties for values of n between 0.15 and 1.00 which covers the practical range for polymer melts. For each velocity iteration cycle the relative change of each internal nodal point velocity was computed and the magnitudes of these changes summed. The convergence criterion used was that this sum should be less than 10^{-5}. Typical numbers of iterations were, for 81 point meshes and an overrelaxation factor of 1.47, from about 20 for $n = 1$ up to about 70 for $n = 0.15$. These figures varied somewhat with π_p, however, and to a lesser extent with S.

Finally the flowrate may be computed from Equation (6) or (12):

$$Q = \Sigma_m \frac{1}{3} \Delta_m (w_i + w_j + w_k) \qquad (26)$$

RESULTS

Some results are presented in three stages; firstly using uniform meshes, then internally modified meshes for improved accuracy, and finally for channels with rounded corners.

Uniform Meshes

Figure 4 shows two typical flow curves for rectangular channels with shape parameters of 0.2. The linear Newtonian ($n = 1$) relationship together with a non-Newtonian ($n = 0.5$, a typical value for polymer melts) curves are shown, the former because an experimentally verified (Squires, 1958; Holmes and Vermeulen, 1968) analytical solution exists (Fenner, 1970) which for $S = 0.2$ is

$$\pi_Q = 0.4457 - 0.07283\, \pi_p \qquad (27)$$

The FD solution for the non-Newtonian curve has also been successfully compared with some experimental results by Fenner (1970).

Considering Newtonian solutions first, Table 1 shows for a range of π_p the true π_Q from Equation (27), followed by the FD solution (Fenner, 1970) for an 81 point mesh and FE solutions for 81, 289, and 441 point uniform meshes, all with an equal number of points in the two coordinate directions. Clearly for the same number of points the FE and FD solutions are virtually identical as was predicted above. As the number of points is increased the FE solution approaches the true (analytical) solution, an essential requirement for any numerical method. While comparisons are restricted here to the flowrate, they can be extended to velocities with the same results as shown by Palit (1969).

Table 2 shows a comparison of the FE and FD non-Newtonian solutions. In this case there is no analytical result so it is assumed that as the mesh is refined the true flowrate is approached. Again for similar meshes the FE and FD solutions are very close.

Modified Meshes

The above FE results for finer meshes can equally well be obtained from a FD analysis. The real power of the FE method lies in its ability to handle nonuniform meshes, with elements concentrated in regions of large velocity gradients. Such regions exist in the top corners of the channel (Figure 1) where the moving and stationary boundaries meet but are of much greater importance in computing the mechanical power applied to the moving boundary than in finding the flowrate (Fenner, 1970).

For relatively shallow channels ($S < 0.5$ say) Purday (1949) showed that for Newtonian flow in rectangular channels with stationary boundaries the velocity profile may be approximated by the product of a quadratic function of y and a similar power-law function of x, with the power approximately equal to the reciprocal of S. This implies that particularly in the x-direction the only significant velocity gradients are near the sides of the channel. With the top boundary moving this condition still applies, but the profile in the y-direction depends on the flowrate. Examination of the present solutions for both Newtonian and non-Newtonian flows confirms these conclusions.

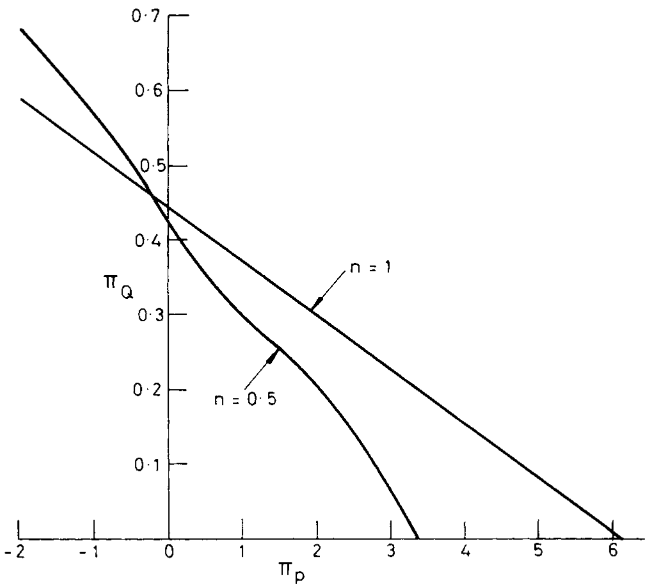

Fig. 4. Dimensionless flow curves for rectangular channels with $S = 0.2$.

Table 1. Newtonian Dimensionless Flowrates for $S = 0.2$ and Uniform Meshes

π_p	True π_Q	FD 81 points	FE 81 points	FE 289 points	FE 441 points
−2	0.5914	0.558	0.562	0.581	0.584
0	0.4457	0.424	0.426	0.439	0.441
2	0.3000	0.289	0.290	0.296	0.297
4	0.1544	0.154	0.154	0.153	0.153
6	0.0087	0.019	0.018	0.010	0.009

Table 2. Non-Newtonian ($n = 0.5$) Dimensionless Flowrates for $S = 0.2$ and Uniform Meshes

π_p	FD 81 points	FE 81 points	FE 289 points	FE 441 points
−2	0.636	0.642	0.668	0.672
−1	0.533	0.537	0.555	0.559
0	0.405	0.407	0.419	0.421
1	0.283	0.285	0.291	0.292
2	0.199	0.199	0.201	0.201
3	0.127	0.120	0.116	0.116
4	0.003	−0.011	−0.024	−0.026

Extensive tests with various meshes and several values of n have shown that there is no significant advantage to be gained from modifying meshes in the y-direction, but in the x-direction the following modification offers the best general improvement in accuracy

$$x' = \tfrac{1}{2}B\ (1 + sgn(2x - B) \cdot |2x/B - 1|^S) \qquad (28)$$

where x is the nodal point coordinate for the uniform mesh, x' its modified value, and sgn means the sign of (that is, for $2x > B$, $sgn\ (2x - B) = +1$ and for $2x < B$, $sgn\ (2x - B) = -1$). This formula was obtained empirically but can be seen to closely reflect Purday's findings. It has not been proved to represent the best possible transformation, but such a transformation would be required for each new problem and would require much more computation than the actual solution. The modified mesh is symmetrical and for $S = 0.2$ the coordinates in the left half of an 81 (9×9) point mesh are changed as follows:

$x/B = 0$	0.125	0.25	0.375	0.5
$x'/B = 0$	0.028	0.065	0.121	0.5

Using the modification formula (28) the results in Tables 3 and 4 are obtained. Comparison with Tables 1 and 2 shows the improvement in accuracy obtained with the modified meshes. Indeed, the results for the 81 point modified mesh are generally superior to those for the 441 point uniform mesh, and very much faster to compute. The only minor exceptions are at low flowrate where the situation is complicated by the solutions tending to the true values from above rather than below.

Modified Channel Geometry

The results presented so far are for rectangular channel geometries to which FD methods are also applicable, including the modified meshes (Gosman et al., 1969). Once the simple geometry is lost, however, the FD method is much more difficult to apply, whereas the FE method is not affected provided the boundary can be approximated by a series of sufficiently short straight lines.

To illustrate the application to nonrectangular geometries a channel with rounded corners (the bottom corners in Figure 1) is considered. Practical examples include extruder screws with rounded fillets between the flight and screw root. The corner radii are H, centered at $y = H$, $x = H$ and $(B - H)$, and the shape parameter is 0.333. The required FE mesh is readily obtained from the rectangular one by moving all the nodal points in the end quadrants towards the centers of curvature by an amount proportional to the displacement of the boundary on the same radius. In terms of computer programming such operations are trivial, and are much simpler than constructing a mesh manually and then reading the data into the computer.

Figure 5 shows equivalent flow curves for rectangular and rounded channels. The expected flowrate reduction with rounded corners is only obtained at small π_p. At high π_p the trend is reversed, a phenomenon which has already been explained for finite rectangular as opposed to infinite width channels by Martin (1969) and Fenner (1970).

DISCUSSION

The results presented here are typical of a FE analysis of slow non-Newtonian channel flow. The method, which

Table 3. Newtonian Dimensionless Flowrates for $S = 0.2$ and Modified Meshes

π_p	True π_Q	FE 81 points	FE 289 points	FE 441 points
−2	0.5914	0.5869	0.5903	0.5907
0	0.4457	0.4443	0.4454	0.4455
2	0.3000	0.3016	0.3005	0.3003
4	0.1544	0.1590	0.1556	0.1551
6	0.0087	0.0164	0.0106	0.0100

Table 4. Non-Newtonian ($n = 0.5$) Dimensionless Flowrates for $S = 0.2$ and Modified Meshes

π_p	FE 81 points	FE 289 points	FE 441 points
−2	0.6798	0.6819	0.6821
−1	0.5637	0.5664	0.5668
0	0.4256	0.4267	0.4269
1	0.2971	0.2961	0.2959
2	0.2078	0.2052	0.2048
3	0.1247	0.1187	0.1180

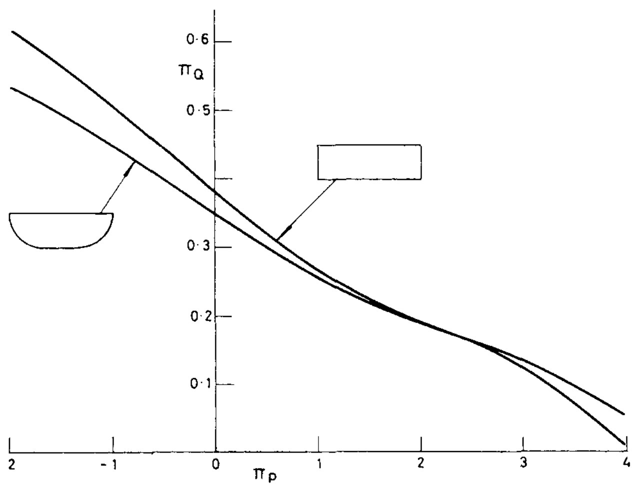

Fig. 5. Dimensionless flow curves for rectangular and rounded channels with $S = 0.333$, $n = 0.5$.

uses triangular elements and assumes a linear velocity distribution over each element, is analogous to the five point FD method both in terms of equations and results. Whereas the main usefulness of FD methods is in simple geometries, FE methods are very flexible in that to a large extent the size, shape, and orientation of the elements relative to the coordinate axes are arbitrary. This allows arbitrary boundary geometry to be handled and the optimum mesh distribution for maximum accuracy. The usefulness of highly nonuniform meshes has only been established here, however, for flows normal to the plane of the mesh. More general flows are more restrictive on mesh variations, just as with FD methods, as discussed by Martin (1969).

FD methods in their basic form use uniform distributions of points arranged in rows parallel to the coordinate axes, resulting in simple difference equations. While in principle FD equations can be constructed for an arbitrary mesh of points (Varga, 1962) such an approach is rarely used in practice. One of the most flexible FD formulations, used by Gosman et al. (1969), is for curvilinear-orthogonal coordinate systems with points distributed (not necessarily uniformly) in rows parallel to the coordinate axes. Such a formulation loses much of the simplicity of the basic FD method but does not allow complete flexibility of boundary shapes and mesh distribution. Indeed, the choice of coordinates is determined by the need to fit the boundaries so that arbitrary or even compound shapes such as the rounded channel are difficult to handle.

FE methods employ points on the boundaries of elements of very flexible proportions, usually triangles or quadrilaterals for two-dimensional problems. Arbitrary problem boundary shapes can be fitted, particularly using triangular elements. The type of Cartesian FE formulation used here can be applied to any problem not normally treated in these coordinates. While linear velocity distributions are assumed here, higher order polynomial functions are possible, but would have to be associated with more complex element shapes or more nodal points per element (see Zienkiewicz, 1971; Atkinson et al., 1969, 1970). Significantly shorter computing times are, however, claimed for such formulations (Oden and Somogyi, 1969). Three point triangular elements are by far the simplest to use, particularly for non-Newtonian flow, and at least for the present application give satisfactory solutions. While attention has been restricted to isothermal flows normal to the problem region, the method can be applied to more general situations, for example, including flow and heat transfer in the plane of the mesh.

NOTATION

a, b = local nodal point coordinates (Figure 2)
B = channel width
C_1, C_2, C_3 = constants in velocity distribution, Equation (13)
$[d]$ = element geometry matrix, Equation (14)
F = rate of dissipation of total potential energy
$[F]$ = dimensionless pressure force matrix
H = channel depth
I_2 = second invariant of rate of deformation tensor
i, j, k = subscripts referencing nodal points
$[K]$ = overall viscous 'stiffness' matrix
$[k^{(m)}]$ = element stiffness matrix
m = subscript or superscript referencing element number
N = total number of nodal points
n = power-law index
P_z = downstream pressure gradient
Q = volumetric flowrate
S = channel shape parameter (H/B)
s_{pi}, s_{vi} = pressure and viscous force components
T = superscript indicating transposed matrix
V_z = downstream velocity of moving boundary
W = dimensionless downstream velocity (w/V_z)
w = downstream velocity
X, Y = dimensionless coordinates $(x/H, y/H)$
x, y, z = Cartesian coordinates (Figure 1)
x' = modified x-coordinate
γ_0 = reference shear rate
Δ = element area
μ = viscosity
μ_0 = effective viscosity at γ_0
π_p = dimensionless downstream pressure gradient $(P_zH/\bar{\tau})$
π_Q = dimensionless flowrate (Q/BHV_z)
τ = viscous shear stress
$\bar{\tau}$ = shear stress at mean channel shear rate
χ = integral of functional F

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